F: Introduction to Bessel Functions

Bessel's equation of order n is the equation

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)y = 0 \quad . \tag{1}$$

Since it is a linear second order differential equation, two linearly independent solutions are the *Bessel functions of first and second kinds*, notationally given by $J_n(x), Y_n(x)$, so the general solution to (1) is $y(x) = C_1J_n(x) + C_2Y_n(x)$. Some properties are:

1. Much about $J_n(x)$ comes from the series expansion

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{k!(n+k)!}\left(\frac{x}{2}\right)^{2k+n} .$$

   For example,

$$J_n(0) = \begin{cases} 1 & n = 0 \\ 0 & n > 0 \end{cases}$$

2. Another consequence of the series representation of $J_n(x)$ are the *shift formulas*:

$$\frac{d}{dx}\left[x^{-n}J_n(x)\right] = -x^{-n}J_{n+1}(x)$$
$$\frac{d}{dx}\left[x^{n}J_n(x)\right] = x^{n}J_{n-1}(x)$$

   For example, $J_0'(x) = -J_1(x)$. These indicate that $J_n(x)$ oscillates: the first shift formula shows $\frac{d}{dx}[x^{-n}J_n(x)]$ must vanish between successive zeros of $x^{-n}J_{n+1}(x)$. That is, the zeros of J_n, J_{n+1} separate each other (see Figure 1).

3. As x increases $J_n(x)$ becomes closer and closer to $\sqrt{\frac{2}{\pi x}}\cos[x - \frac{\pi}{4}(1+2n)]$, that is, like cosine with an (n dependent) phase shift, and an amplitude that decays like $1/\sqrt{x}$. A way we would write this statement of fact is that $J_n(x) \sim \sqrt{\frac{2}{\pi x}}\cos[x - \frac{\pi}{4}(1+2n)]$ as $x \to \infty$. Similarly, $Y_n(x) \sim \sqrt{\frac{2}{\pi x}}\sin[x - \frac{\pi}{4}(1+2n)]$ as $x \to \infty$.

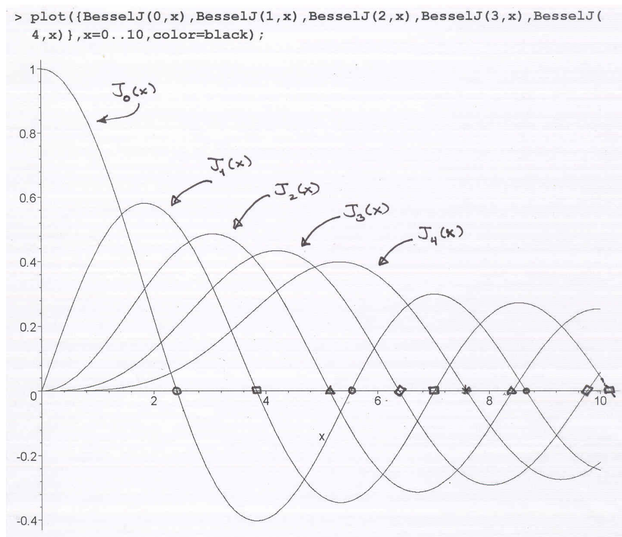

Figure 1: The first five Bessel functions of the first kind.

4. As $x \to 0$, $J_n(x)$ remains bounded (see Figure 1), but $Y_n(x)$ goes **unbounded** as $x \to 0$. See, for example, Figure 2. Put another way,

$$Y_0(x) \sim \ln(x) \cdot \{\text{power series in } x\} \qquad x \to 0$$

while for $n > 0$

$$Y_n(x) \sim \frac{1}{x^n} \cdot \{\text{power series in } x\} \qquad x \to 0$$

Therefore, for our diffusion problem (or a vibration problem) in the disk, $Y_n(x)$ is not of physical significance for us.

5. A consequence of the above properties, with, for each fixed n, $\left\{\lambda_{nk}, J_n(\sqrt{\lambda_{nk}}r)\right\}_{k=1}^{\infty}$ being the set of eigenvalue-eigenfunction pairs, then the orthogonality relation for $J_n(x)$ is given by

$$\int_0^a J_n(\sqrt{\lambda_{nj}}r)J_n(\sqrt{\lambda_{nk}}r)\, rdr = \begin{cases} \frac{a^2}{2}\left(J_n'(\sqrt{\lambda_{nk}}a)\right)^2 & j = k \\ 0 & j \neq k \end{cases}$$

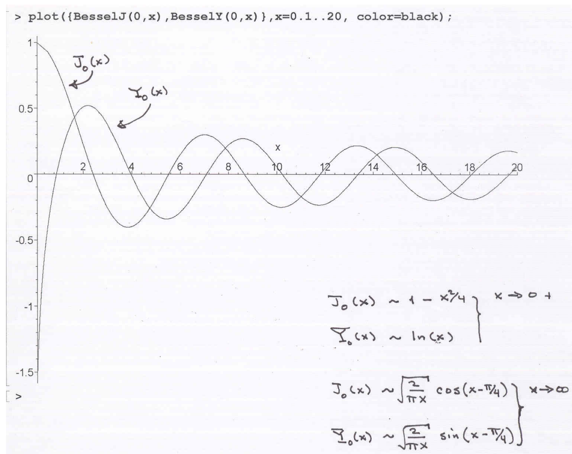

Figure 2: Graph of $J_0(x)$ and $Y_0(x)$ and what they look like for small and large argument.

In these Notes we are introduced to the order zero equation first, namely

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + x^2y = 0 \tag{2}$$

For the diffusion problem on the disk, example 2, Section 18, we have solution $\phi(r) = C_1 J_0(\sqrt{\lambda}r)$. Thus, we must have $J_0(\sqrt{\lambda}a) = 0$.

From Figure 2 it is clear that J_0 (and Y_0) oscillate, so there is an infinite number of zeros for J_0, $J_0(s) = 0 \quad \rightarrow \quad 0 < s_1 < s_2 < s_3 < \ldots$, which implies $\lambda_k = (s_k/a)^2$ for $k = 1, 2, \ldots$ being the required eigenvalues for the disk problem. There is no neat formula for the zeros of J_0, but they are tabulated in various tables, and easily estimated using various software packages like Maple, Mathematica, Matlab, MathCad, etc. For example, in Maple one would use the operator BesselJZeros(0,k,…,m) to generate a sequence of zeros of $J_0(x)$ from the kth to the mth (inclusive) zero.

For the disk problem, with eigenvalues $\lambda_n = (s_n/a)^2$, and associated eigenfunctions $\phi_n(r) = J_0(\sqrt{\lambda_n}r)$, we have $T(t) = T_n(t) = e^{-\lambda_n Dt}$, so the solution to that problem has the form

$$u(r,t) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n Dt} J_0(\sqrt{\lambda_n}r) \quad . \tag{3}$$

Hence, letting $t \to 0$, we write

$$f(r) = \sum_{n=1}^{\infty} a_n J_0(\sqrt{\lambda_n} r) \ . \tag{4}$$

Now (4) is often called a **Bessel-Fourier series** for $f(r)$. To complete the problem we need to multiply both sides of (4) by $\sigma(r)\phi_m(r) = J_0(\sqrt{\lambda_m} r)r$ and integrate:

$$\int_0^a f(r) J_0(\sqrt{\lambda_m} r) r dr = \sum_{n=1}^{\infty} a_n \int_0^a J_0(\sqrt{\lambda_n} r) J_0(\sqrt{\lambda_m} r) r dr \ \ .$$

This might look a bit more complicated than when we were dealing with sines and cosines, but our procedure has remained unaltered. From the orthogonality condition above, we have

$$\int_0^a f(r) J_0(\sqrt{\lambda_m} r) r dr = a_m \int_0^a J_0^2(\sqrt{\lambda_m} r) r dr \ ;$$

that is,

$$a_m = \frac{\int_0^a f(r) J_0(\sqrt{\lambda_m} r) r dr}{\int_0^a J_0^2(\sqrt{\lambda_m} r) r dr} = \frac{\int_0^a f(r) J_0(\sqrt{\lambda_m} r) r dr}{(a^2/2)(J_0'(\sqrt{\lambda_m} a))^2} = \frac{2}{a^2} \frac{\int_0^a f(r) J_0(\sqrt{\lambda_m} r) r dr}{(J_1(\sqrt{\lambda_m} a))^2} \ .$$

Remark: There is a slight inconsistency in the notation here regarding the eigenvalues. Since we mainly deal with J_0, for order $n = 0$, its mth eigenvalue is written λ_m rather than λ_{0m}.

Remark: There are a large number of **special functions**, besides the Bessel functions, which satisfy differential equations, and come from solving partial differential equations. A few examples of equations are:

1. **Legendre**: $\frac{d}{dx}((1-x^2)\frac{d\phi}{dx}) + \lambda\phi = 0 \ , \ |x| < 1 \ .$
2. **Tchebycheff**: $\frac{d}{dx}(\sqrt{1-x^2}\frac{d\phi}{dx}) + \lambda(1-x^2)^{-1/2}\phi = 0 \ , \ |x| < 1 \ .$
3. **Hermite**: $\frac{d^2v}{dx^2} + (1-x^2)v + \lambda v = 0 \ , \ |x| < \infty$. Here $v = \phi e^{-x^2/2}$, where $\phi = H_n(x)$ is a Hermite polynomial; then ϕ solves the equation $\frac{d}{dx}(e^{-x^2}\frac{d\phi}{dx}) + \lambda e^{-x^2}\phi = 0 \ , \ |x| < \infty \ .$

4. **Laguerre**: $\frac{d}{dx}(xe^{-x}\frac{d\phi}{dx}) + \lambda e^{-x}\phi = 0 \;\;,\;\; x > 0 \;\;.$

Exercises:
Notice that equation $xy'' + y' + xy = 0$ is (2), that is, Bessel's equation of order zero, and that $xy'' + y' + x^{-1}y = 0$ is a Cauchy-Euler equation. What about the equation $xy'' + y' + y = 0$?

1. Show that the general solution to the equation

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 0$$

is $y(x) = C_1 J_0(2\sqrt{x}) + C_2 Y_0(2\sqrt{x})$. (Let $y(x) = f(z)$, where $z = \sqrt{x}$, and obtain the equation for f.)

2. A hanging chain of length l undergoes small oscillations in the plane. Assuming that tensile force in the chain does not differ appreciably from that required to withstand gravity, the governing equation is

$$g\frac{\partial}{\partial x}\left(x\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial t^2}$$

for small lateral displacements $u(x,t)$, where x is measured upward from the free end of the chain, and g is the constant acceleration of gravity. Assume $u(x,0) = f(x), u_t(x,0) = 0$. Obtain the series solution for u.[1] (The transformation $z = \sqrt{x}$ and the exercise above will be useful.)

3. Suppose we reconsider the hanging chain problem of part 2, but now, instead of a fixed end at $x = l$, we are able to shake this end at the ceiling periodically, say $u(l,t) = A\cos(\omega t)$. If we look for a solution of the form $u(x,t) = U(x)\cos(\omega t)$, find $U(x)$.[2]

[1] This problem came from from the book *Partial differential Equations, Theory and Technique* by Carrier and Pearson.
[2] This problem was given to me by R. Rostamian.